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Exponential distributions in a mechanical model for earthquakes

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We study statistical distributions in a mechanical model for an earthquake fault introduced by Burridge and Knopoff [R. Burridge and L. Knopoff, Bull. Seismol. Soc. Am. **57**, 341 (1967)]. Our investigations on the size (moment), time duration, and number of blocks involved in an event show that exponential distributions are found in a given range of the parameter space. This occurs when the two kinds of springs present in the model have the same, or approximately the same, value for the elastic constants. Exponential distributions were also seen recently in an experimental system to model earthquakelike dynamics and in faults of the earth's crust. [S1063-651X(96)00612-5]

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I. INTRODUCTION

Systems that present stick-slip dynamics and scaling invariance have attracted considerable attention recently [1,2]. This was triggered, in part, by a seminal paper of Bak, Tang, and Wiesenfeld, which showed that a class of systems presenting avalanches or earthquakelike dynamics can naturally attain a critical state characterized by power-law distributions [1]. They denoted this phenomenon self-organized criticality.

Recently, two experimental studies in a continuous elastic system have been performed to model the stick-slip dynamics observed in earthquakelike phenomena. In one of them [3], a glass rod is pulled on top of a latex membrane. In this system the slipping events appear as detachment waves and a wide distribution of event sizes was observed. However, robust scaling behavior was not seen.

In the other system, studied by Rubio and Galeano [4], the elastic medium consists of a transparent gel sheared between two coaxial circular cylinders. The inner cylinder is rotated at very low angular speed. Thus this system closely resembles the physics of a spring-block model [5] with periodic boundary conditions. Due to the friction between the gel and the cylinders, stick-slip behavior is seen in the form of detachment waves. Rubio and Galeano were able to identify four regimes in this system, by varying the gel's rigidity, the rotor angular speed, and the friction properties of the gel with the inner cylinder. The regimes identified were: (a) uniform slipping of the gel with respect to the rotor; (b) relaxation events, in which a big event starts to quickly propagate through the whole cell in an almost periodic way; (c) nearly periodic regimes of a solitonlike character, and (d) regimes

involving events with many sizes and time scales. In the latter case, they found exponential statistical distributions in amplitude, duration, and separation time between events.

With respect to theoretical and numerical studies, one of the systems used to model the dynamics of earthquakes has been the model introduced by Burridge and Knopoff (BK) in 1967, and investigated extensively in recent publications [6,7]. Although most of the studies on the Burridge-Knopoff model have been concentrated on the parameter region where a partial power law (partial here means of limited size) of the distributions of event sizes is found, this regime in fact has not been seen yet in experimental studies of a homogeneous system. The three first regimes (a), (b), and (c) found by Rubio and Galeano have been observed in numerical studies of the BK model. No reports so far exist of exponential distributions in the BK model.

In this paper we show that exponential distributions are also seen in the BK model. This kind of distribution is observed when the two kinds of springs in the model have the same, or approximately the same, value for the elastic constants. They are present in quantities such as the time duration, number of blocks displaced, and total displacement (moment) of the blocks involved in an event. The BK model, with all the springs having the same elastic constant, models a system in which the coupling between neighboring blocks has the same magnitude as the coupling between the blocks and the plate to which they are attached. Therefore, this is a model for a system that is homogeneous. It does, in fact, reproduce the results found in controlled experimental studies [4] of a homogeneous system that presents stick-skip dynamics.

With respect to the earth's crust, exponential distributions were observed at least in two studies. Cowie *et al.* [8] found that the statistics of the lengths, scarp heights, and spacing of normal faults on the flanks of a given midocean ridge (the East Pacific Rise, between 19°S and 21°N) obey exponential distributions. The same kind of distribution was observed by

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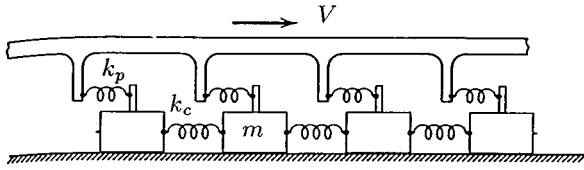
FIG. 1. System studied, which consists of a chain of N blocks connected by linear springs of constant k_c. The blocks are on a flat surface and each block is connected to a bar through a spring of constant k_p. The bar is pulled with constant velocity V. Between the surface and the blocks there is a velocity weakening friction force F.

Brooks and Allmendinger [9] in the study of the fault spatial distribution in El Teniente Mine, in Central Chile. Consequently, not only fractal distributions (e.g., the Gutenberg-Richter law) have been seen in studies of the earth's crust. Thus, as in the BK model, different distributions of fault properties on the earth can be found, depending on the features of the considered region.

The paper is organized as follows: in Sec. II we describe the model; in the third section we study the statistical distributions and the different regimes that appear when the spring constants are varied; the last section is dedicated to the conclusions.

II. BURRIDGE-KNOPOFF MODEL

The homogeneous version of the Burridge and Knopoff model is shown in Fig. 1. It consists of a one dimensional array of N blocks, each of mass m, coupled by springs of constant k_c to one another, and by a spring of constant k_p to a rigid pulling bar that moves at constant velocity V. In equilibrium, when all the springs are unstretched, adjacent blocks are separated by a distance a. The blocks rest upon a stationary surface, which provides a frictional force that impedes the motion of the blocks. In the version considered in Ref. [6] the friction is a decreasing function of the velocity, the same for all blocks. The equation of motion for the jth mass when it is moving is

$$m\ddot{X}_j = k_c(X_{j+1} - 2X_j + X_{j-1}) - k_p(X_j - Vt) - F(\dot{X}_j/V_f), \quad (1)$$

where X_j denotes the displacement of the block measured with respect to the position where the sum of the elastic forces on it is zero. The last term in Eq. (1) represents the nonlinear and velocity dependent frictional force, which is given by

$$F(\dot{X}_j/V_f) = \begin{cases} F_0/(1 + \dot{X}_j/V_f), & \text{if } \dot{X}_j > 0 \\ -\infty, & \text{if } \dot{X}_j < 0, \end{cases}$$

where V_f is the characteristic velocity for the friction. Here, we do not allow backward motions and the friction force gets arbitrarily high to avoid it. This is just for computational convenience, and this does not change the main results presented here.

As shown in Ref. [7], this model has five independent velocity scales. Two appear explicitly in Eq. (1): (i) the pull-

ing velocity V and (ii) the characteristic velocity V_f. Two more can be defined in terms of the spring constants, the mass and F_0: (iii) $V_0 = F_0/\sqrt{mk_p}$ and (iv) $V_l = (F_0/k_p)\sqrt{k_c/m}$. Here, V_0 and V_l correspond to the maximum velocities of a single block held by a spring of constant k_p and k_c, respectively, when it has been displaced by the characteristic distance F_0/k_p in the absence of friction. The fifth velocity is the sound velocity $V_s = a\sqrt{k_c/m}$, which depends on the equilibrium spacing a of the blocks and does not appear explicitly in Eq. (1). Depending on the relative sizes of these velocities, we would expect a different behavior for the system.

Following Ref. [6], we introduce different dimensionless variables

$$U_j = \frac{k_p}{F_0}X_j, \quad \tau = (m/k_p)^{1/2}t, \quad (2)$$

so that Eq. (1) can be written in dimensionless form

$$\ddot{U}_j = \nu_l^2(U_{j+1} - 2U_j + U_{j-1}) - U_j + \nu\tau - \Phi(\dot{U}_j/\nu_f), \quad (3)$$

where

$$\Phi(\dot{U}_j/\nu_f) = \begin{cases} 1/(1 + \dot{U}_j/\nu_f), & \text{if } \dot{U}_j > 0 \\ -\infty, & \text{if } \dot{U}_j < 0. \end{cases}$$

Now, $\nu_l = \sqrt{k_c/k_p}$, $\nu = V/V_0$, and $\nu_f = V_f/V_0$. Dots here represent derivatives with respect to the scaled time τ. The sound velocity becomes $\nu_s = a'\sqrt{k_c/k_p} = a'\nu_l$, where $a' \equiv ak_p/k_c$. Without losing generality we can take the dimensionless length $a' \equiv 1$. Thus the sound velocity ν_s becomes identical to ν_l. The five velocities V, V_f, V_l, V_s, and V_0 have been transformed, respectively, into $\nu, \nu_f, \nu_l, \nu_s \equiv \nu_l$, and $\nu_0 \equiv 1$. Consequently, the model has three relevant parameters, ν, ν_f, and $\nu_l \equiv \nu_s$, which completely determine the behavior of the system. In the case of an open boundary condition, which is the one we consider here, we have $U_0 = U_1$ and $U_{N+1} = U_N$.

For fixed ν_l and ν (large ν_l and small ν), a transition was reported in Ref. [7] when two velocities cross, namely, when $\nu_f = \nu_0 \equiv 1$. For $\nu_f = \infty$ one finds that the motion of the system is continuous. No block ever stops. As ν_f decreases, one sees small regions of stationary blocks. When ν_f becomes less than 1 these stationary (event-free) regions begin to percolate across the entire system. The motion of the fault in this small ν_f region now occurs in abrupt large events.

Here we report another transition, which occurs when the velocity ν_l is varied for fixed values of ν and ν_f. We find that for ν_l less than a given value the event sizes and event durations obey exponential distributions. Another value is observed for ν_l, above which the distributions present partial power laws, namely, events that have a size smaller than a critical value have power-law distributions, and events that are larger than this value obey different statistics. In the intermediate regime for ν_l, the distributions are neither power laws nor exponentials. As ν_l gets close to zero, the exponential distributions disappear, since in that limit the blocks are disconnected, and only one-block events are observed. Evi-
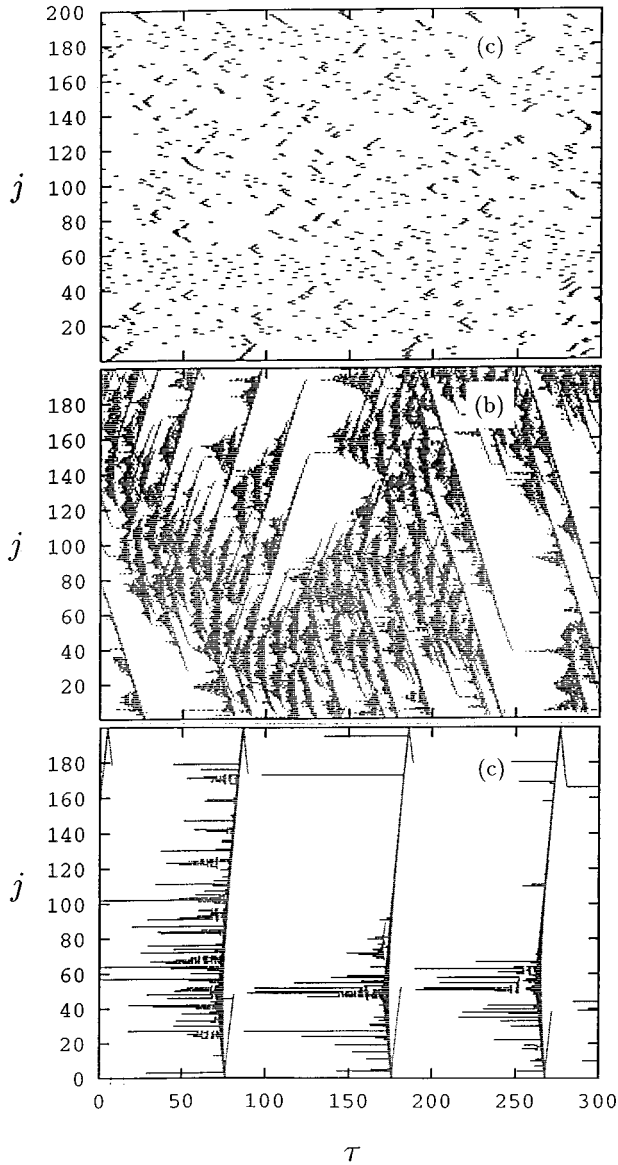
FIG. 2. Projections of the block velocities $\dot{U}_j$ onto the $j-\tau$ plane for a 200 block system with $\nu=0.01$, $\nu_f=1/6$, and (a) $\nu_l=1$, (b) $\nu_l=3$, and (c) $\nu_l=10$.
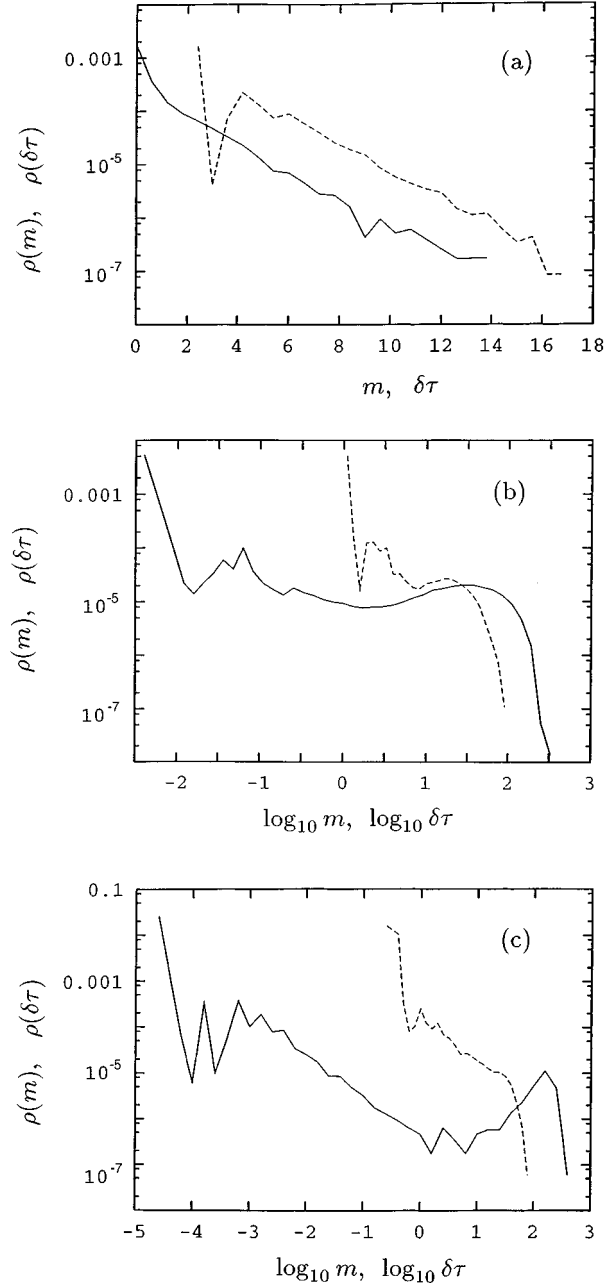
FIG. 3. Statistical distribution $\rho(m)$ of the moment of the events (solid lines) and the distribution $\rho(\delta\tau)$ of the time durations of the events (dashed lines) for (a) $\nu_l=1$, (b) $\nu_l=3$, and (c) $\nu_l=10$. In all cases, $N=300$, $\nu=0.01$, and $\nu_f=1/6$.

dence of a different behavior when ν_l is varied was found in Ref. [10]. However, no statistical distributions were calculated in that paper.

In the numerical simulations of this letter we have generally started the system with the blocks at rest and with the sum of the elastic forces in each block equal to zero. Other initial conditions were considered and the results did not change. We have also considered periodic boundary conditions, and the results were the same (except for the few events involving the boundaries of the chain). Before we start to compile statistics we let the system evolve until it reaches a statistical stationary state. Here we fix the pulling velocity and the characteristic velocity to the following values: $\nu=0.01$ and $\nu_f=1/6$. It is beyond the scope of this paper to do a detailed study on the effects of variations in the pulling velocity and in the characteristic speed. Studies on event distributions for varying ν_f and ν with fixed ν_l can be found in [6]. We expect that the kind of transition we report

here will be observed for other values of ν_f and ν, and the value of ν_l where the transition occurs probably varies with the characteristic and pulling velocities.

III. NUMERICAL RESULTS

To illustrate the different behaviors of the system when ν_l is varied, we show in Fig. 2 projections of the block velocities $\dot{U}_j$ onto the $j-\tau$ plane. A black dot in the figure means that the j block is moving at time τ and a white one, that it is at rest. We show three different cases with $\nu_l=1,3,10$. It is clear that a transition occurs as ν_l is changed. For small ν_l, we see clusters of moving blocks,

which involve only a small number of masses. As ν_l is increased, one finds clusters involving a small number of blocks, as well as clusters involving a large number of masses. These large ruptures appear $\vee$ shaped and travel with the sound speed ν_l.

Now we investigate the distributions of moment and time duration for different values of ν_l. The moment of an earthquake is a measure of its size, and in dimensionless units it is defined as $m = \Sigma_j \delta U_j$, where δU_j is the displacement of the jth block and the sum is over the blocks displaced during the event. The distribution $\rho(m)$ is the number of events with magnitude m divided by the number of blocks N in the chain and by the total time of dynamical evolution. For $\nu_l \lesssim 2$ we find that $\rho(m)$ is governed by an exponential function, namely, $\rho(m) \sim \exp(m/m_c)$, where m_c is a characteristic moment. The result of $\rho(m)$ with $\nu_l = 1$ is shown in Fig. 3(a) as a solid line. We have also found that the characteristic moment decreases as ν_l becomes smaller, since when $\nu_l = 0$ the blocks become disconnected and only events involving one block are observed. For intermediate values of ν_l ($2 \lesssim \nu_l \lesssim 4$) ones sees a regime for $\rho(m)$ which is neither power-law nor exponential behavior. This is shown in Fig. 3(b) where the solid line represents $\rho(m)$ for $\nu_l = 3$. For large ν_l ($\nu_l \gtrsim 4$) one sees the appearance of a partial power-law distribution. This was the parameter region investigated by Carlson and Langer [6]. Beyond the power-law cutoff one sees a bump in the distributions, which increases in height as ν_l becomes larger. We have found that the events that do not belong to the scaling region become more and more frequent as ν_l is increased. Thus, in the limit of very large ν_l the power law tends to disappear. In Fig. 3(c) the solid line represents $\rho(m)$ for $\nu_l = 10$.

For the distributions of time durations $\delta\tau$ of the events we find qualitatively the same kind of behavior as found for the distributions of moments. For $\nu_l \lesssim 2$ we get an exponential function, $\rho(\delta\tau) \sim \exp(\delta\tau/\delta\tau_c)$. This is shown in Fig. 3(a) for $\nu_l = 1$ as a dashed line. Again, we have found that the characteristic time duration $\delta\tau_c$ decreases as ν_l gets smaller. For $\nu_l \gtrsim 4$ one sees partial power-law distributions, as shown in Fig. 3(c) for $\nu_l = 10$ as a dashed line. For intermediate values of ν_l ($2 \lesssim \nu_l \lesssim 4$) the distributions are neither exponential nor power-law ones, as we show in Fig. 3(b) for $\nu_l = 3$ (dashed line).

We have also studied the statistical distributions $\rho(n)$ of an event involving n blocks. Again, the same kind of behavior described above is seen, namely, exponential distributions for $\nu_l \lesssim 2$, partial power laws for $\nu_l \gtrsim 4$, and a regime without any scaling features for intermediate values of ν_l.

IV. CONCLUSIONS

We have found exponential distributions for the sizes and time durations of the events in the Burridge-Knopoff model for earthquakes when the two kinds of springs have the same (or approximately the same) elastic constants. In this situation the velocity ν_l and the sound velocity are equal (or approximately equal) to one. For intermediate values of ν_l, the distributions are neither exponential nor power law. For large ν_l, partial power laws (namely, a scaling region of limited size) are observed. Since in the experimental system studied by Rubio and Galeano [4] the elastic gel is homogeneous, this implies that the exponential distributions they found are consistent with our results for the case in which the springs of the BK model have identical elastic constants. The study here helps to understand why different fault distributions can be found in different locations of the earth's crust.

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